Spatially Growing Disturbances in Liquid Films

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In their recent study of the liquid film instability, Krantz and Owens (1973) find that the amplification rate for temporally growing disturbances obtained by Anshus and Goren (1966), after being modified by Gaster's (1962) transformation, does not agree well with the observation of Krantz and Goren (1971). On the other hand, a similar solution of the Orr-Sommerfeld equation for spatially growing disturbances is found to agree closely with the observed amplification rate. Although the comparison on the wave speed is not as good, Krantz and Owens conclude that the spatial formulation of the Orr-Sommerfeld equation is superior for liquid film flows. However, Lin (1975) points out that such a conclusion is reached through an inappropriate use of Gaster's theorem. He also points out that the temporal solution and the spatial solution being related by Gaster's transformation are equivalent within the framework of linear theory. In response to this criticism, Krantz (1975) uses the quotation of Gaster: "This method which involves an expansion is only valid for small values of amplification (or damping) and for other flows which are less stable than boundary layers it is necessary to solve the characteristic equation for each type of mode." He contends that the solution for the spatial case cannot be obtained from that for the temporal case, since, according to Krantz, the falling film is highly unstable.

In this note we solve the eigenvalue problem for spatially growing long waves in a film. We then show that theoretical results thus obtained are identical to those obtained from the known temporal solution by use of Gaster's transformation. This contradicts Krantz's contention. We also take this opportunity to answer a frequently raised question: "Is the nonlinear stability theory developed by Lin (1969, 1970, 1971, 1974) and others for temporally growing disturbances relevant to film flows in which the disturbance is observed to grow spatially?"

During the initial stage, the instability of a liquid film due to spatially growing disturbances is governed by the Orr-Sommerfeld equation (compare Agrawal, 1972)

$$\phi^{IV} - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha R_e[U(\phi'' - \alpha^2) - U''\phi] - iR_e\omega(\phi'' - \alpha^2\phi) \quad (1)$$

where U is the primary flow normalized by the average velocity U_a, $R_e \equiv U_a d/\nu$ is the Reynolds number, d and ν being respectively the film thickness and the kinematic viscosity of the fluid, and the amplitude function ϕ is related to the stream function of the disturbances ψ by

$$\psi(x, y, t) = \phi(y) \exp [i(\alpha x - \omega t)]$$

in which x and y are the distances measured in the unit of d respectively in the direction of the flow and in a direction normal to the flow from the free surface into the liquid film, t the time normalized by d/U_a, α the dimensionless wave number, and ω is the dimensionless wave frequency. The above equation implies that the disturbance under consideration is two dimensional. For spatially growing disturbances $\alpha = \alpha_r + i\alpha_i$ where $\alpha_r = 2\pi d/\lambda$ (λ being the wave length) is the wave number and α_i is the spatial amplification rate or damping rate depending on if $\alpha_i < 0$ or $\alpha_i > 0$. ω is the frequency defined by

$$\omega = \frac{2\pi d}{\lambda} \frac{C_r}{U_a}$$

where C_r is the wave speed. The boundary conditions for this problem are (compare Agrawal, 1972)

$$\phi'(1) = 0$$

$$\phi(1) = 0$$

$$\phi''(0) + \left(\alpha^2 - \frac{3\alpha}{\omega - 3\alpha/2}\right)\phi(0) = 0$$

$$[\alpha^2(3 \cot \beta + \alpha^2 W_e R_e)/(\omega - 3\alpha/2)]\phi(0) + [R_e(\omega - 3\alpha/2) + 3i\alpha^2]\phi'(0) - i\phi'''(0) = 0 \quad (2)$$

where β is the angle of inclination, $W_e = T/\rho d U_a^2$ is the Weber number, T and ρ being, respectively, the surface tension and the fluid density. For long gravity waves observed in the liquid film $d << \lambda$ and $C_r = O(U_a)$ and thus $\omega << 1$. We expand the solution in powers of ω, that is,

$$\phi = \phi_0 + \omega\phi_1 + \omega^2\phi_2 + \ldots$$

$$\alpha = \alpha_0 + \omega\alpha_1 + \omega^2\alpha_2 + \ldots$$

Substituting the above expansion into (1) and (2) and solving the resulting equations by the method of regular perturbation, we obtain

$$\begin{aligned}\phi = (1 - 2y + y^2) &+ \frac{i}{3}\omega R_e\left[\left(\frac{7}{20} - \frac{Q}{9}\right)y + \left(\frac{2Q}{9} - \frac{4}{5}\right)y^2 + \left(\frac{1}{2} - \frac{Q}{9}\right)y^3 - \frac{1}{20}y^5\right] \\ &+ \frac{1}{9}(\omega R_e)^2\left[A_2 y + B_2 y^2 + C_2 y^3 + \frac{1}{6}R_e^{-2}y^4 + \frac{7}{800}y^5 + \left(1 - \frac{Q}{9}\right)\frac{y^7}{560} - \frac{y^9}{2240}\right] + O[(\omega R_e)^3]\end{aligned} \quad (3)$$

and

$$\begin{aligned}\alpha = \alpha_r + i\alpha_i = \frac{\omega}{3} &+ \frac{i}{9}\omega^2 R_e\left[\frac{Q}{9} - \frac{2}{5}\right] \\ &+ \frac{1}{27}\omega^3 R_e^2\left[R^{-2} - \frac{2Q^2}{81} + \frac{2Q}{105} + \frac{44}{175} + \frac{4}{9}\omega^2 W_e\left(\frac{1}{45} - \frac{Q}{162}\right)\right]\end{aligned} \quad (4)$$

where

$$Q = (3 \cot \beta/R_e + \omega^2 W_e/9)$$

$$A_2 = \frac{Q^2}{27} - \frac{2}{3} R_e^{-2} - \frac{23Q}{210} - \frac{901}{11200} + \frac{4}{9}\omega^2 W_e \left(\frac{Q}{162} - \frac{1}{45}\right)$$

$$B_2 = \frac{3}{2} R_e^{-2} - \frac{2Q^2}{27} + \frac{32}{175} + \frac{68Q}{315} + \frac{4}{9}\omega^2 W_e \left(\frac{2}{45} - \frac{Q}{81}\right)$$

$$C_2 = \frac{Q^2}{27} - \frac{19Q}{180} - R_e^{-2} - \frac{9}{80} + \frac{4}{9}\omega^2 W_e \left(\frac{Q}{162} - \frac{1}{45}\right)$$

Note that the above results are valid even if $R_e >> 1$ as long as $\omega R_e << 1$. It follows from (4) that the spatial amplification is given by

$$\alpha_i = \frac{1}{9}\omega^2 R_e (Q/9 - 2/5) \qquad (5)$$

We now show that the above results can also be obtained directly from the results of the temporal case by use of Gaster's transformation. The growth rate β_i and the frequency ω for the temporal case are obtained by Benjamin (1957) and Yih (1963) and are given by

$$\begin{aligned} \beta_i &= \alpha_r^2 R_e (6/5 - Q'/3) \\ \omega &= \alpha_r[3 + (\alpha_r R_e)^2(-12/7 + 10Q'/21) - 3\alpha_r^2] \end{aligned} \qquad (6)$$

where $Q' = 3 \cot \beta/R_e + \alpha_r^2 W_e$. According to Gaster's theorem, this is related to α_i by

$$\beta_i = -\alpha_i \frac{\partial \omega}{\partial \alpha_r} + O[\alpha_i^2 (\beta_i)_{\max.}]$$

Substituting (6) into the above equation, we have

$$\alpha_i = \alpha_r^2 R_e (Q'/9 - 2/5) + O(\alpha_r^4) \qquad (7)$$

It is obvious from the second equation of (6) that

$$\alpha_r = \omega/3 + O(\omega^3)$$

Hence, (7) can be rewritten as

$$\alpha_i = \frac{\omega^2}{9} R_e (Q/9 - 2/5) + O(\alpha_r^4)$$

which is identical to (5) obtained from the eigenvalue problem for the spatial disturbances. This completes the demonstration that the temporal and spatial formulation of the Orr-Sommerfeld equation for the liquid film flow are equivalent for long gravity waves.

The nonlinear evolution of the initial disturbances which grow spatially with the amplification rate given by (5) has been studied by Agrawal (1972). He developed a spatial formulation of the Landau-Stuart approach. The second Landau coefficient for spatially growing disturbances is obtained. The nonlinear evolution of spatially growing disturbances is found to be qualitatively the same as that of temporally growing disturbances studied by Lin (1969, 1970, 1971, 1974). In particular, both types of disturbance are found to approach almost the same supercritically stable wave forms. However, no simple relation similar to the Gaster transformation is found for the second Landau coefficients for the two different types of disturbances.

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NOTATION

C_r = wave speed
d = film thickness
R_e = Reynolds number = $U_a d/\nu$
T = surface tension
t = dimensionless time
U = primary flow velocity
U_a = average velocity
W_e = Weber number = $T/\rho d U_a^2$
x, y = Cartesian coordinate

Greek Letters

α = complex wave speed = $\alpha_r + i\alpha_i$
α_r = wave number = $2\pi d/\lambda$
α_i = negative of spatial amplification rate
β = angle of inclination
β_i = temporal amplification rate
ω = wave frequency
λ = wave length
ϕ = amplitude function
ψ = disturbance streamfunction
ν = kinematic viscosity
ρ = fluid density

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